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Rational expectations, information and asset markets: an introduction

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Abstract

In this chapter, the author provides, in easily accessible language, a description of the ways in which uncertainty affects prices in financial markets. The model which the author formulates contains two stages, in the first of which agents make their production decisions. Subsequently, agents enter the financial market, where supply and demand for futures contracts are equated.

The futures contract is a financial asset, whose gross return is the futures price. In a stochastic version of the model, and under the assumptions that dealers are well-informed and risk-neutral, it can be demonstrated that the futures price is equal to the present discounted value of the expected spot price at contract termination.

The equilibrium may not exist if the dealers have diverse information: i.e. if some dealers are well-informed and others are less so. In this situation, there will be different beliefs about the expected spot price of the futures contract; but these differences may be resolved if the prices themselves are recognised as conveying information. the author proves that if this supplementary information is aggregated perfectly, the futures market becomes informationally efficient and equilibrium is attained.

The solution/concept employed is that of perfect foresight, the deterministic version of the rational expectations equilibrium. This concept is discussed in the final sections of the chapter, where the focus is on risk aversion and on the assumption underlying rational expectations – that dealers have correct beliefs about the joint probability distribution of the futures price, the spot price and private information.

1. Introduction

Financial markets are a subject of perpetual fascination to economists and others. There are very large sums of money to be gained and lost on them. They are obviously crucially important not only to the people and institutions who invest directly, but also to many others who invest indirectly through holding unit trusts (mutual funds), pension or life assurance policies. Moreover, the financial markets do not operate in isolation; they affect and are affected by the rest of the economy.

One important economic function of such markets is the spreading and sharing of risk. An entrepreneur can reduce the risks which he carries by selling shares in his firm. Investors may be willing to carry some of the risk because they are less risk averse than the entrepreneur. They may also be willing to invest even if they are more risk averse

[1] I am grateful to Craig Alexander, Jeremy Edwards, Anna Lemessany, Peter Sinclair and Martin Weale for comments on an earlier version of this paper.

because the market allows them to hold a diversified portfolio which reduces risk. Investing £10,000 in ten different firms whose profits are imperfectly correlated is very much less risky than investing £10,000 in one of the firms. The view that such markets perform a socially important function in spreading risk reasonably well is widely held (see Arrow (1964) and Diamond (1967) for theoretical models). But there are distinguished dissidents; in Chapter 12 of the General Theory, Keynes argues forcefully that the markets increasingly provide a casino for speculators, rather than a guide for investors, and may be socially useless or even positively dangerous.

Recent theoretical work on asset markets, based on the rational expectations hypothesis, has argued that they may have an additional informational role. Traders have information which affects their evaluation of the value of assets, the demand for the assets, and thus prices. Other traders may attempt to infer the information from prices. The major achievement of recent work has been to develop a coherent description of this phenomenon, and use it to ask how well the markets transmit and aggregate the information.

Much of this literature is highly technical, and inaccessible without a considerable mathematical apparatus. Yet the basic issues can be understood with much less background, as this paper seeks to demonstrate. It is written as an introduction to recent work on information in asset markets, assuming intermediate microeconomics, enough calculus to differentiate a quadratic, a little manipulation of linear equations, and enough probability theory to know about means, variances, and conditional distributions. I use expected utility theory, but anyone who does not know the theory, and is willing to take on trust my assertion that it is a sensible way to model choice under uncertainty, should be able to follow the argument.

Much of the paper is concerned with elaborating a simple model. The model introduced in §2 is the standard deterministic partial equilibrium model of supply and demand in a spot market, modified by the assumption that production decisions must be made before the market operates on the basis of price expectations. I use this model to introduce a perfect foresight equilibrium; the deterministic version of a rational expectations equilibrium. In §3 I introduce a futures market, operating at the date when production decisions are made. A futures contract is a financial asset, whose gross return is the spot price. I argue that arbitrage implies that in this deterministic model, if expectations are held with certainty, the futures price must be equal to the present discounted value of the expected spot price. Section 4 introduces briefly the expected utility theory of choice under uncertainty. Section 5 applies this theory to a stochastic version of the model on the assumption that dealers are risk neutral, using an arbitrage argument to establish that the futures price is equal to the present discounted value of the expected spot price. Section 6 shows how the simple arbitrage argument breaks down when risk neutral dealers have diverse information, introducing the information role of asset prices. The formal definition of a rational expectations equilibrium in an asset market with asymmetric information is introduced in §7. Section 8 introduces risk aversion, simplifying matters mathematically by working with exponential utility functions, and normal random variables. The joint equilibrium of the spot and futures market when dealers are risk averse is calculated, on the assumption that no one has any private information about the spot price when trading on the futures market. Information is introduced in §9, firstly on the assumption that all dealers have the same information, secondly on the assumption that there are informed and uninformed traders, but the informed traders all have the same information, and thirdly on the assumption that dealers have diverse information. In this model the futures market is remarkably informationally efficient; it aggregates information pefectly. Section 10 is concerned with the implications and robustness of the informational efficiency result in this and related models. In the models which I use, calculating the rational expectations equilibrium is relatively straightforward, but in §11 I introduce a version of the spot and futures market model which has no rational expectations equilibrium. I discuss the nature and significance of the problems associated with the existence of rational expectations equilibrium, and the literature on the subject. Section 12 attempts an evaluation of the models, discussing the assumptions, concentrating largely on the rational expectations assumption, and referring briefly to the empirical and experimental evidence. Section 13 discusses some open questions prompted by these models.

The results which I establish have no claims to originality, the first model which I develop has its origins in the cobweb model (Kaldor 1934), and in Muth's paper on rational expectations (1961). The futures market model is based on Danthine (1978), and related to Grossman (1976 and 1977) and Bray (1981). The non-existence example in §11 is new in detail, but is similar to that of Kreps (1977). I give references to other, related literature, where appropriate. A more technical introduction to this and many other topics can be found in Radner's (1982) survey of 'Equilibrium under uncertainty' and in the symposium issue (April 1982) of the *Journal of Economic Theory* on 'Rational expectations in microeconomic models', in particular the introduction by Jordan and Radner. Stiglitz (1982) discusses a range of issues concerned with information and capital markets.

2. Supply and demand with a production lag: perfect foresight equilibrium

In the standard model of supply and demand, production and consumption decisions are taken simultaneously, based on the price. If production takes time, production decisions have to be based on the expected price. For example, a farmer plants a crop in January which will be harvested and sold in June. To begin with, assume that there is no uncertainty, an assumption which will be relaxed in §4. Demand $D(p_s)$ is a deterministic function of p_s, the spot price of wheat in June. Supply $S[p_s^e]$ is a deterministic function of p_s^e, the farmers' point expectation belief in January about what the spot price will be in June. For now, assume that all farmers are subjectively certain about what the price will be, and all have the same beliefs. If the market in June clears, supply equals demand. $D[p_s] = S[p_s]$. The expected price determines production which in turn determines the actual price. In fact, the price p_s is a function of the expected price.

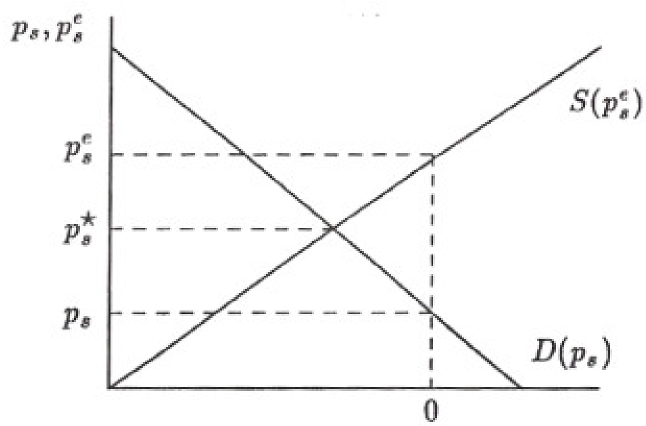

Figure 1

In Fig. 1, when the price is p_s^e, $Q = S[p_s^e]$ is produced. When Q is put on the spot market in June the price is p_s. If $p_s \neq p_s^e$, the farmers, despite their subjective certainty, are wrong. Beliefs are wrong unless $p_s^e = p_s^\star$, the price at which the supply and demand curves intersect so $S[p_s^\star] = D[p_s^\star]$. This could well be described as a self-fulfilling belief. However, the standard terminology is a perfect foresight equilibrium or, more recently (following Muth 1961), a rational expectations equilibrium.

A rational expectations equilibrium can be defined as a situation in which people do not make systematic mistakes in forecasting. In this case, where beliefs are point expectations held with certainty, rational expectations equilibrium requires that beliefs be correct, i.e. that people have perfect foresight. The rational expectations assumption is now used very widely, but remains controversial. The assumption avoids many of the difficult dynamic problems apparently associated with expectation formation, making it possible to proceed with other questions. For the time being I will simply assume rational expectations without further discussion, returning to the matter in §12.

3. Financing production: Futures markets and arbitrage

The revenue from selling the crop arrives some time after most of the production costs are incurred. This leaves a farmer with the problem of finding funds to cover the investment in planting the crop. He may have sufficient wealth to finance this from his own resources. If not he will have to borrow.

Assume that everyone knows that the price in June will be p_s. There are perfect capital markets, that is, the farmer can borrow or lend as much as he wishes at the same interest rate. £ borrowed in January must be repaid with £$(1+r)$ in June. Suppose that a farmer has wealth W_0 in January, and incurs the costs of producing output y, which have a present value in January of $C(y)$. He invests the remainder of his wealth $W_0 - C(y)$ at interest rate r until June. His wealth in June is the sum of his revenue from output $p_s y$ and the return on his other investment

$$W = p_s y + (W_0 - C(y))(1+r) = p_s y - C(y)(1+r) + W_0(1+r).$$

The value of profits from production in June is $p_s y - C(y)(1+r)$. Note that $W_0 - C(y)$ may be negative, in which case the farmer is borrowing to cover some of his costs. The farmer maximizes his June wealth by maximizing profits. If C is a convex function of y and $p_s > C'(0)(1+r)$, this is done by setting $p_s = C'(y)(1+r)$. The value of y is independent of his initial wealth, which simply determines how much, if anything, he has to borrow.

The farmer may also finance his production by selling on the futures market. A futures market is an institution on which money is exchanged for promises to deliver goods in the future. For example, a farmer may sell wheat in January for delivery in June. As before, suppose the farmer has wealth W_0 in January, produces y, incurring costs $C(y)$, and sells z on the futures market at price p_f. This leaves him $W_0 - C(y) + p_f z$ to invest at interest r. In June he sells the remainder of his output $y - z$ on the spot market. His wealth in June is

$$\begin{aligned} W &= p_s(y - z) + (W_0 - C(y) + p_f z)(1 + r) \\ &= p_s y - C(y)(1 + r) + (p_f(1 + r) - p_s)z + W_0(1 + r). \quad (3.1) \end{aligned}$$

The farmer maximizes his wealth, as before, by choosing output y so $p_s = C'(y)(1 + r)$. If $p_f > p_s/(1 + r)$, so that the futures price exceeds the present discounted value of the spot price, he can make arbitrarily large profits by selling on the futures market. He will increase z indefinitely, and will wish to set $z > y$, selling more on the futures market than he produces, meeting the shortfall $z - y$ by buying on the spot market. However, he is unlikely to find a willing buyer at this price. There are two possible classes of buyers, consumers and speculators. Consumers (e.g. food manufacturers and wholesalers) may choose to buy futures in January rather than waiting to buy on the spot market in June, thus hedging against uncertainty about the June spot price. For the sake of simplicity I will assume that consumers do not participate in the futures market; if they did it would complicate the models without substantially affecting the conclusions. Speculators buy futures contracts, which they sell on the spot market, never actually taking delivery of the goods, in the hope of making a profit on the difference between the futures price and the present value of the spot price. Suppose a speculator with wealth W_0 in January buys x futures contracts in January, sells x on the spot market in June, and invests the rest of his wealth in the safe asset paying interest r. His wealth will be

$$\begin{aligned} W &= p_s x + (W_0 - p_f x)(1 + r) \\ &= (p_s - p_f(1 + r)x + W_0(1 + r). \qquad (3.2) \end{aligned}$$

If $p_f > p_s/(1 + r)$, both speculators and farmers will wish to sell futures. With no willing buyers the market cannot clear. If $p_f < p_s/(1 + r)$, both speculators and farmers will want to buy futures. Thus the only price at which the futures market can clear is when $p_f = p_s/(1 + r)$. This is an example of an arbitrage argument—these arguments are based on the premise that in equilibrium it cannot be possible for anyone to make arbitrarily large certain profits. If the market is perfectly arbitraged $p_s = p_f/(1 + r)$. The wealth in June of farmers and speculators does not depend on the size of their future trades. In this deterministic model with perfect foresight, a futures contract is a safe asset paying interest r. There is no reason for anyone to use the futures market in preference to borrowing or lending at rate r elsewhere. If the futures market ceased to exist no one would be any better or worse off.

In fact, under uncertainty there seems little reason for the futures market to exist. Any understanding of futures markets, and other asset markets such as the stock market, depends upon introducing uncertainty.

4. Choice under uncertainty

The farmer takes risks in both the quantity and price of output. A futures market allows the farmer to shift the price risks to speculators. If his output y is certain, he can completely eliminate the risk by setting $z = y$, selling his entire output on the futures market. But why will the speculator be willing to assume the risk, and at what price? The currently available answers to this, and many other questions about economics under uncertainty, are derived from a widely accepted model of choice under uncertainty: the theory of expected utility. An introduction to the theory can be found in, among other places, Deaton and Muellbauer (1980), in a survey by Schoemaker (1982) or, in a valuable collection of readings, Diamond and Rothschild (1978).

Assume that an investor has decided to invest a certain amount W_0 for a period. He has a number of different assets to choose between, and a definite set of beliefs about the joint profitability distribution of the returns on the different assets. He cares only about the probability distribution of his wealth $\widetilde{W}$ at the end of the period, which depends upon the way he allocates his initial wealth W_0 between the different assets. The theory of expected utility shows that if his preferences over the probability distribution of $\widetilde{W}$ satisfy some plausible assumptions, he will choose a portfolio which maximizes the mathematical expectation $EU(\widetilde{W})$ of a function $U(\widetilde{W})$, given his beliefs about the probabilities. For a discrete probability distribution $EU(\widetilde{W}) = \sum_i U(W_i)p_i$, where W_i is wealth in state i and p_i the probability of state i. For a continuous probability distribution

$$EU(\widetilde{W}) = \int_{-\infty}^{\infty} U(W)f(W)dW ,$$

where f is the probability density function. In both cases the probability distribution depends upon the investor's beliefs, and his choice of portfolio.

The theory has two essential elements, the utility function and the probability distribution which determines the mathematical expectation. The functional form of the utility function U describes attitudes to risk. U is increasing provided investors prefer more to less wealth. If $U(\widetilde{W}) = (\widetilde{W})$ the investor is risk neutral, caring only about expected wealth, and not at all about its riskiness. If $U(\widetilde{W})$ is strictly concave the investor is risk averse, strictly preferring investments yielding the expectation of $\widetilde{W}$ for sure, to random $\widetilde{W}$. Risk aversion in investment choices for an individual seems highly plausible, and is often assumed.

The assumption that uncertainty can be described in terms of probability distribution is widely made today, but historically has not commanded universal acceptance. Keynes was a notable dissenter. There is very little controversy about applying the mathematical theory of probability to assess the probabilities associated with a series of similar events, where after a time there is enough data to construct probabilities from frequency distributions (for example, weather or life expectancy data), situations described by Knight (1921) as risk. The argument is rather whether meaningful probabilities can be assigned to unique events, where there is no objective frequency data to rely on, situations described by Knight as uncertainty. The subjectivist or Bayesian viewpont on probability is that Knight's distinction is invalid. It is always possible to elicit probabilities by forcing people to make bets (see Raiffa 1968). There is, however, no guarantee in subjectivist theory that different people will form the same probability distributions, unless there is frequency data to base them on, which brings us back to Knight's risk. For some purposes it is enough to assume that people act as if they had subjective beliefs expressible as probability distributions. However, many models postulate that people have the same correct beliefs about probability distributions (rational expectations). These models do not seem to be applicable to situations which Knight would describe as uncertain.

I am now in a position to use the theory of expected utility to extend the theory of asset pricing under certainty to uncertainty. Initially I will assume risk neutrality and then proceed to consider risk aversion.

5. Risk neutrality: Arbitrage again

Returning to the futures market example, suppose that once farmers have chosen their level of inputs their output y is certain. The June spot price is uncertain because spot demand is uncertain. A risk neutral farmer will choose his output y and futures sales z to maximize the expected value of his wealth; from (3.1) this is

$$E\widetilde{W} = E\widetilde{p}_s y - C(y)(1+r) + (p_f(1+r) - E\widetilde{p}_s)z + W_0(1+r)\,.$$

(Throughout this paper a tilde ~ above a variable indicates that it is random.) A speculator will choose his futures purchases x to maximize the expected value of his wealth; from (3.2) this is

$$E\widetilde{W} = (E\widetilde{p}_s - p_f(1+r))x + W_0(1+r)\,.$$

Decisions depend upon the mathematical expectation $E\widetilde{p}_s$ of $\widetilde{p}_s$, its average value. The risk neutral dealers do not care about any other characteristics of the probability distribution. $E\widetilde{p}_s$ is not a point expectation held with certainty; the dealers are aware that there is uncertainty and would expect to observe that usually $E\widetilde{p}_s \neq \widetilde{p}_s$.

Precisely the same arbitrage argument as before implies that unless $p_f = E\widetilde{p}_s/(1+r)$ there are unlimited positive expected profits to be made and the market cannot clear. The argument is less compelling than under certainty. Although a speculator may wish to exploit opportunities for making positive expected profits, he may not be able to do so. Suppose that $E\widetilde{p}_s > p_f(1+r)$, so buying futures contracts generates a positive expected return. A risk neutral speculator will choose to spend his entire wealth on futures contracts, he will also wish to borrow without limit to exploit further the opportunity for profit. There is a chance that the spot price will be so low that he cannot repay his debts; lending to the speculator becomes risky. Speculators may face either a higher interest rate than r, or limits on credit, limiting their ability to arbitrage the market.

6. Diverse information

The simple arbitrage argument also breaks down if different dealers (farmers and speculators) have different beliefs about the expected spot price. This is not incompatible with the dealers having rational expectations, if they have access to different information. Suppose, for example, that $\widetilde{p}_s = \widetilde{I} + \widetilde{e}$ where $\widetilde{I}$ and $\widetilde{e}$ are independent random variables, $E\widetilde{e} = 0$, and so $E\widetilde{p}_s = E\widetilde{I}$. There are two types of dealers. The informed dealers observe $\widetilde{I}$ before the futures market opens; their expectation of $\widetilde{p}_s$ is conditional upon $\widetilde{I}$, $E[\widetilde{p}_s|\widetilde{I}] = \widetilde{I}$. The uninformed dealers observe nothing, their expectation of $\widetilde{p}_s$ is $E\widetilde{p}_s = E\widetilde{I}$. If both types of dealers are risk neutral, face no borrowing constraints, and stick to their beliefs, the informed will want to buy or sell an unlimited amount unless $p_f = E[\widetilde{p}_s|\widetilde{I}]/(1+r) = \widetilde{I}/(1+r)$, and the uninformed dealers will want to buy or sell an unlimited amount unless $\widetilde{p}_f = E\widetilde{p}_s/(1+r)$. Unless by coincidence $E[\widetilde{p}_s|\widetilde{I}] = E\widetilde{p}_s$ (i.e. if $\widetilde{I} = E\widetilde{I}$), the market apparently cannot clear.

It is, however, most unlikely that the uninformed dealers will stick to their beliefs. Knowing that there are informed dealers in the market whose trading affects the futures price they will try to make inferences from the futures price about the spot price. They are using the price of a financial asset, a futures contract, to make judgements about its quality. Judging quality from price is not confined to financial markets. Consumers may also do so, assuming that cheap goods are also cheap and nasty. One of the major successes of recent economic theory has been the development of models which take this into account.

In these models prices have two roles, their conventional role in determining budget sets for consumers and profit opportunities for firms, and an additional role in transmitting information. Hayek (1945) in a discussion of decentralization and planning argues that the conventional role of prices must also be understood as an informational one. In standard Walrasian competitive equilibrium models, once households and firms know current prices they have no use for any further information about the plans, characteristics and opportunities of others in the economy, they need make no attempt to infer this information from prices. As Grossman (1981) argues, recent models of asymmetric information move beyond this; some agents want some information held by others, in this case information about the spot price in the future. They try to infer as much information as they can from current prices. In some cases the price system may be entirely efficient at transmitting information; prices are so informative that there is no additional information currently known to anyone in the economy which would be helpful. In other cases prices may be less informationally efficient, conveying some information, but still leaving a frustrated desire to see the current contents of someone else's mind, or computer file. In either case agents are trying to look beyond prices, to solve an inference problem, which is unnecessary in standard Walrasian models. The central question addressed by the models which I am about to discuss is how informationally efficient are prices? These models make use of a rational expectations equilibrium. I will now show how this this equilibrium is defined, and explain how it yields an equilibrium price for this example.

7. Rational expectations equilibrium and risk neutrality

The definition of a rational expectations equilibrium for the spot and futures market has four parts. A very similar definition can be formulated for any asset market model. The first part describes how dealers form their beliefs.

Part 1. *Each dealer (farmer or speculator) observes some private information $\tilde{I}_i$ and the futures price $\tilde{p}_f$. Given this information he has beliefs about the spot price $\tilde{p}_s$ which can be expressed as a conditional probability distribution.*

For example, dealer i might believe that given the futures price $\tilde{p}_f$ and private information $\tilde{I}_i$, the conditional distribution of $\tilde{p}_s$ was normal with mean $E[\tilde{p}_s|\tilde{p}_f, \tilde{I}_i] = \frac{1}{2}\tilde{p}_f + \frac{1}{4}\tilde{I}_i$ and variance $\frac{1}{8}$. At this stage I have not required that the beliefs be correct, only that they exist.

The second part of the definition states that given their beliefs dealers choose their portfolio in accordance with expected utility theory.

Part 2. *Each dealer chooses the holding of futures contracts, and for farmers, output, which maximizes his expected utility given his beliefs about the spot price, conditional upon his private information and the futures price.*

Parts 1 and 2 of the definition give the supply and demand for futures. Note that supply and demand are affected by both the numerical value of the futures price and information, and by beliefs. If a risk neutral dealer believes that $E[\tilde{p}_s|\tilde{p}_f, \tilde{I}_i] = \frac{1}{2}\tilde{p}_f + \frac{1}{4}\tilde{I}_i$, he will buy or sell an unlimited amount depending on whether $\tilde{p}_f[\frac{1}{2}\tilde{p}_f + \frac{1}{4}\tilde{I}_i]/(1+r)$ is positive or negative. To emphasize this point I will write $d_i[\tilde{p}_f, \tilde{I}_i; B_i]$ for dealer i's demand for futures, where B_i is shorthand for beliefs.

The next part of the definition is

Part 3. *The spot and futures prices are at levels where both markets clear.*

In different years the information will be different, so if the markets are to clear prices must be a function of the information. Demand and the market clearing prices also depend upon beliefs, so I will write

$$\tilde{p}_f = f[\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n; B_1, B_2, B_n]$$

$$\tilde{p}_s = g[\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n; B_1, B_2, B_n]$$

An omniscient economist could calculate the function f. Knowing the joint distribution of $[\tilde{I}_1, \tilde{I}_2, \tilde{I}_n]$ the economist could then calculate the joint distribution of $[\tilde{p}_f, \tilde{p}_s, \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n]$, and so the conditional distribution of $\tilde{p}_s$ given $\tilde{p}_f$ and $\tilde{I}_i$ for each i. This would tell the economist what the correct beliefs for each dealer would be, call them $\tilde{B}_i$. As the joint distributions depend upon the original beliefs,

$[B_1, B_2, \ldots, B_n]$, the correct beliefs $[\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_n]$ also depend upon the original beliefs. A more formal way of saying the same thing is that $[\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_n]$ is a function of $[B_1, B_2, \ldots, B_n]$.

The last part of the definition is

Part 4. *Each agent has rational expectations. They have correct beliefs about the joint probability distribution of the futures price, spot price and private information, so*

$$B_i = \widetilde{B}_i \qquad i = 1, 2, \ldots, n\,.$$

Note that this states that beliefs about the entire conditional probability distribution are correct. Much of the macroeconomic literature works with models where only the conditional mean is relevant, but the rational expectations hypothesis is not confined to such models.

This definition may appear unnecessarily long winded. Stating that the beliefs are correct in Part 1 would make for greater brevity, but stating the definition in this way gives more insight. It is helpful in calculating the rational expectations equilibrium in simple models, where making a guess about the functional form of beliefs, calculating supply and demand, and then checking to see if there is indeed a set of beliefs which generates rational expectations often works. This approach is also very helpful in understanding issues associated with the existence and stability of rational expectations equilibrium.

I have stated the definition in terms of a spot and futures market, but very similar definitions can be formulated for any set of financial asset markets. I have not been specific about the information $\widetilde{I}_i$. All that is required is that it be a random variable, but it may be continuous or discrete, a scalar or a vector. It may always take the same value, $\widetilde{I}_i = 0$, in which case it is effectively no information.

I will now calculate the rational expectations equilibrium for the futures market example with risk neutral dealers. Here the informed agents observe $\widetilde{I}$ and the uninformed agents observe nothing. Recall that $\widetilde{p}_s = \widetilde{I} + \widetilde{e}$, $\widetilde{I}$ and $\widetilde{e}$ are independent and $E\widetilde{e} = 0$. In accordance with Part 1 of the definition, suppose that the informed dealers believe that $E[\widetilde{p}_s|\widetilde{p}_f] = \lambda\widetilde{p}_f$ where λ is a constant. Utility maximization (Part 2 of the definition) for risk neutral dealers implies that the informed dealers will want to buy or sell an unlimited amount unless $E[\widetilde{p}_s|\widetilde{I}, \widetilde{p}_f] = \widetilde{p}_f(1 + r)$, and the uninformed dealers will want to buy or sell an unlimited amount unless $E[\widetilde{p}_s|\widetilde{p}_f] = \widetilde{p}_f(1 + r)$. Thus market clearing (Part 3 of the definition) implies that

$$E[\widetilde{p}_s|\widetilde{I}, \widetilde{p}_f] = \widetilde{I} = \widetilde{p}_f(1 + r)$$

and

$$E[\tilde{p}_s|\tilde{p}_f] = \lambda\tilde{p}_f = \tilde{p}_f(1+r)\,.$$

This is impossible unless $\lambda = 1+r$, and $\tilde{p}_f = \tilde{I}/(1+r)$. It remains to check that Part 4 of the definition holds. If $\tilde{p}_f = \tilde{I}/(1+r)$, knowing $\tilde{p}_f$ tells the informed dealers nothing about $\tilde{I}$ and $\tilde{p}_s$ which they did not know already from observing $\tilde{I}$ directly. As $\tilde{p}_s = \tilde{I} + \tilde{e}$, the correct conditional expectation for the informed dealers is $E[\tilde{p}_s|\tilde{I}] = E[\tilde{p}_s|\tilde{I}, \tilde{p}_f] = \tilde{I}$. The uninformed dealers observe $\tilde{p}_f = \tilde{I}(1+r)$, so can infer $\tilde{I}$ from $\tilde{p}_f$, knowing that $E[\tilde{p}_s|\tilde{I}] = \tilde{I}$, their correct conditional expectation is $E[\tilde{p}_s|\tilde{I}] = E[\tilde{p}_s|\tilde{p}_f] = \tilde{I} = (1+r)\tilde{p}_f$, which is the form assumed with $\lambda = 1+r$. This is a rational expectations equilibrium.

This is a very striking result, indicating that the market is completely efficient as a transmitter of information from the informed to the uninformed. Much of the recent theoretical work on asset markets has been concerned with investigating the circumstances under which a rational expectations equilibrium exists, and is informationally efficient.

This example has a number of peculiar features. The assumption of risk neutrality is special, and I have argued that even with risk neutrality the market may not be perfectly arbitraged. In equilibrium neither farmers nor speculators have any reason to trade futures. The expected profits from trade are always zero. It seems possible that the futures market will die away. But without a futures market the informational differences will persist, so there will be a motive for trade. These peculiarities stem from the risk neutrality assumption.

8. Rational expectations equilibrium under risk aversion

I will now introduce risk aversion into the model. This can generate considerable mathematical complexities, which I will minimize by assuming that both farmers and speculators have utility functions of the form

$$U_i(\widetilde{W}) = -e^{k_i\widetilde{W}} \equiv \exp(-k_i\widetilde{W})$$

where k_i is a positive constant. I will use the second form of notation, which avoids the need for superscripts. Remember that 'exp' is an abbreviation for 'exponential' and not for 'expectation'.

This utility function is widely used and has some attractive properties. Its first derivative is positive ($U' > 0$) implying that utility is increasing in wealth. The second derivative is negative ($U'' < 0$) implying risk aversion. The constant $k_i = -U''/U'$ is the coefficient of absolute risk aversion, higher values of k_i imply greater risk aversion.

Above all there is the very useful result that if $\widetilde{W}$ is normal with mean $E\widetilde{W}$ and variance var $\widetilde{W}$

$$E\{-\exp(-k\widetilde{W})\} = -\exp(-k[E\widetilde{W} - \frac{1}{2}k \text{ var } \widetilde{W}]). \tag{8.1}$$

This result implies that the expected utility maximizing portfolio is one that maximizes $E\widetilde{W} - \frac{1}{2}k$ var $\widetilde{W}$. As I will demonstrate this makes for a very tractable model of asset demand, which is linear in expected asset return and prices. The major unattractive feature of the utility function, which I will also demonstrate is that asset demand is independent of wealth.[2]

I will now use (8.1) to analyse the behaviour of the spot and futures market model under risk aversion. The first step in defining and calculating the rational expectations equilibrium is a description of the information and beliefs. The first case I will look at is where dealers have no private information, and then at versions with diverse private information. Once the mathematics has been done for the first case the others follow very simply.

Equation (8.2) gives the beliefs described in Part 1 of the definitions of a rational expectations equilibrium. I will use this to derive the utility maximizing speculators' demand for futures, and the farmers' demand for futures and spot supply (Part 2 of the definition). I will then make an assumption about spot demand which enables me to write down market clearing conditions for the spot and futures markets (Part 3 of the definition). These conditions will generate a 'correct distribution' for the spot price which will depend upon the parameters of the model, including μ and σ^2. I will show that there are values of μ and σ^2 which generate correct beliefs (Part 4 of the definition), thus deriving the rational expectations equilibrium.

There are n dealers, m farmers and $n - m$ speculators. Farmers are indexed by $i = 1, 2, \ldots, m$, and speculators by $i = m + 1, \ldots, n$.

Speculators

Speculator 1 has a utility function $-\exp[-k_i\widetilde{W}_i]$. If he buys x_i futures at price p_f, sells them on the spot market at price $\widetilde{p}_s$, gets interest r on a safe asset, and has initial wealth W_{i0}, his final wealth $\widetilde{W}_i$ is from (3.2) a random variable

$$\widetilde{W}_i = (\widetilde{p}_s - \widetilde{p}_f(1 + r))x_i + W_{i0}(1 + r).$$

[2] $E(\exp(-k\widetilde{W}))$ is the moment generating function of the random variable $\widetilde{W}$, an object which mathematicians find interesting. The result is proved in most texts on probability, e.g. Meyer (1970).

As speculators believe that $\widetilde{p}_s \sim N(\mu, \sigma^2)$, they believe that $\widetilde{W}_i$ is normal, and

$$E\widetilde{W}_i = (\mu - p_f(1+r))x_i + W_{i0}(1+r)$$

$$\text{var}\ \widetilde{W}_i = \sigma^2 x_i^2.$$

From (8.1) the speculator will choose x_i to maximize

$$E\widetilde{W}_i - \frac{1}{2}k_i \text{ var } \widetilde{W}_i = (\mu - p_f(1+r))x_i + W_{i0}(1+r) - \frac{1}{2}k_i\sigma^2 x_i^2.$$

Thus

$$x_i = \frac{1}{k_i\sigma^2}(\mu - p_f(1+r)) \qquad (i = 1, \ldots, n). \tag{8.3}$$

The speculator buys futures if $\mu > p_{f(1+r)}$, there is a positive expected profit to be made on holding futures if $\mu < p_{f(1+r)}$, there is an expected loss to be made on holding futures. His trades are inversely proportional to σ^2, the variance of the spot price, and to k_1, which measures risk aversion. Note that x_i does not depend on initial wealth W_{i0}, due to the special utility function for which the coefficient of absolute risk aversion $k_i = -U''/U'$ does not depend on wealth.

Farmers

The speculators choose to take on risk by entering the futures market. If the farmers' output is certain they can entirely avoid risk by hedging; selling their entire output on the futures market. If they sell more or less than this they are assuming risk which they could avoid, in pursuit of profits, effectively acting as speculators. If y_i is farmer i's output, and z_i his future sales, $x_i = y_i - z_i$ can be thought of as speculative purchases of futures. The farmer's wealth is from (3.1) a random variable

$$\widetilde{W}_i = (p_f y_i - C(y_i))(1+r) + (\widetilde{p}_s - p_f(1+r))x_i + W_{i0}(1+r).$$

The first term is profits from production if all output is sold on the futures market. The second term is profits from speculation. The third term is the future value of initial wealth. As he believes that $\widetilde{p}_s \sim N(\mu, \sigma)$, he believes that $\widetilde{W}_i$ is normal, with mean and variance

$$E\widetilde{W}_i = (p_f y_i - C(y_i))(1+r) + (\mu - p_f(1+r))x_i + W_{i0}(1+r)$$

$$\text{var}\ \widetilde{W}_i = \sigma^2 x_i^2.$$

If the farmer has a utility function $-\exp(-k_i\widetilde{W}_i)$ from (8.1) he chooses (x_i, y_i) to maximize

$$E\widetilde{W} - \tfrac{1}{2}k \operatorname{var}\widetilde{W} = (p_f y_i - C(y_i))(1+r) + (\mu - p_f(1+r))x_i + W_{i0}(1+r) - \tfrac{1}{2}\sigma^2 x_i^2.$$

I will assume that the farmer's costs are

$$C(y_i) = \tfrac{1}{2}cy_i^2$$

where c is a positive constant. Thus the farmer will maximize

$$(p_f y_i - \tfrac{1}{2}cy_i^2)(1+r) + (\mu - p_f(1+r))x_i + W_{i0}(1+r) - \tfrac{1}{2}\sigma_i^2 x_i^2.$$

The first order condition for y_i implies that $p_f = cy_i$. The futures price determines the level of output, which is set so that the futures price is equal to the marginal cost of production. This result is valid for arbitrary utility functions. In this case it implies that

$$y_i = c^{-1}p_f \qquad (i = 1, \ldots, m). \tag{8.4}$$

The first order condition for x_i implies that

$$x_i = \frac{1}{k_i\sigma^2}(\mu - p_f(1+r)) \qquad (i = 1, \ldots, m). \tag{8.5}$$

The farmer's speculative demand for futures is precisely the same as if he were a pure speculator. This result is not valid if output is uncertain, but is convenient (see Bray 1981).

The futures market

The futures market clearing condition is

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{m} y_i. \tag{8.6}$$

The sum of speculative demand for futures from farmers and speculators is equal to farmers' output, sold forward to hedge against uncertainty. Using the expressions for x_i and Y_i (8.3)–(8.5)

$$\sum_{i=1}^{n} \frac{1}{k_i\sigma^2}(\mu - p_f(1+r)) = \sum_{i=1}^{m} c^{-1}p_f = mc^{-1}p_f. \tag{8.7}$$

Thus the futures price depends upon the distribution of the spot price μ and σ^2. However, the spot price depends upon the physical quantity produced, $\sum_{i=1}^{m} y_i$, which in turn depends upon the futures price. The equilibria [line missing]

The spot market

Assumption. *Spot demand is*

$$D(\widetilde{p}_s) = \widetilde{a} - b\widetilde{p}_s$$

where $\widetilde{a}$ is a normal random variable with mean $E\widetilde{a}$ and variance var $\widetilde{a}$, and b a positive constant.

Thus spot demand is subject to random variation as $\widetilde{a}$ varies. Spot supply comes from two sources. Farmers sell any output which they have not already sold on the futures market, so farmer i sells spot $x_i = y_i - z_i$. Speculator i sells spot everything which he bought on the futures market from farmers, x_i. Total spot sales $\sum_{i=1}^{n} x_i$ are thus equal to farmers' total output $\sum_{i=1}^{m} y_i$. (This is implied by the futures market clearing condition (8.6)). As (8.4) implies that $\sum_{i=1}^{m} y_i = mc^{-1}p_f$ the spot market clears when

$$\widetilde{a} - b\widetilde{p}_s = mc^{-1}p_f. \tag{8.8}$$

Rational expectations equilibrium

Eliminating p_f from the market clearing conditions (8.7) and (8.8) implies that

$$\widetilde{p}_s = b^{-1}\widetilde{a} - b^{-1}mc^{-1}\phi^{-1}\mu \tag{8.9}$$

where

$$\phi = 1 + r + mc^{-1}\sigma^2\left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1}. \tag{8.10}$$

As $\widetilde{a}$ is normal and all the other terms on the right hand side of (8.9) are constants, $\widetilde{p}_s$ is normal. The dealers' beliefs about the form of the distribution of $\widetilde{p}_s$ is correct. From (8.2) they believe that $\widetilde{p}_s \sim N(\mu, \sigma^2)$. Equation (8.9) implies that

$$E\widetilde{p}_s = b^{-1}E\widetilde{a} - b^{-1}mc^{-1}\phi^{-1}\mu \tag{8.11}$$

and

$$\text{var } \widetilde{p}_s = b^{-2} \text{ var } \widetilde{a}. \tag{8.12}$$

Beliefs about the mean and variance are correct if $E\widetilde{p}_s = \mu$ and var $\widetilde{p}_s = \sigma^2$. In this case (8.11) and (8.12) imply that the beliefs are correct if and only

$$\sigma^2 = b^{-1} \text{ var } \widetilde{a} \tag{8.13}$$

$$E\widetilde{p}_s = \theta^{-1}\phi E\widetilde{a} \tag{8.14}$$

where substituting for σ^2 in (8.10)

$$\phi = 1 + r + mc^{-1}b^{-2} \text{ var } \tilde{a}\left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1}. \tag{8.15}$$

and

$$\theta = b\phi + mc^{-1}. \tag{8.16}$$

Thus from (8.9), (8.14) and (8.16) as $\mu = E\tilde{p}_s$

$$\tilde{p}_s = \theta^{-1}\phi E\tilde{a} + b^{-1}(\tilde{a} - E\tilde{a}) \tag{8.17}$$

and from (8.8) and (8.17)

$$p_f = \theta^{-1}E\tilde{a}. \tag{8.18}$$

If the futures price is given by (8.18) and dealers' beliefs about the expected spot price by (8.13)–(8.16) the futures market clears. The futures price determines output. Output determines the distribution of the spot price. At this futures price, and this expected spot price, dealers' beliefs about the distribution of the spot price are correct. This is a rational expectations equilibrium.

Introducing risk aversion changes the model in several respects. If all dealers are risk neutral, arbitrage implies that $(1 + r)p_f = E\tilde{p}_s$: the expected return on risky futures is the same as the return on the safe asset. Dealers are indifferent about how many futures they hold, and have no positive reason to trade on the futures market. In this model with risk aversion (8.14) and (8.18) imply that $\phi p_f = E\tilde{p}_s$, and from (8.15) $\phi > 1 + r$. The risk premium $\phi - (1 + r)$ is an increasing function of the variance of the spot price b^{-2} var $\tilde{a}$, and each dealer's risk aversion parameter k_i. Speculators are willing to take on some of the farmer's risk in order to earn a positive expected return. This model in fact overemphasizes the riskiness of speculative portfolios, because it considers only a single risky asset. In practice speculators can diminish, but not eliminate, risk by holding a portfolio of several risky assets whose returns are imperfectly correlated.

Both speculators and farmers wish to hold definite amounts of futures, and the market will trade actively. As $\mu = E\tilde{p}_s > p_f(1+r)$, (8.3) and (8.5) imply that demand from speculators, and the speculative element of farmers' demand will be strictly positive in equilibrium. Farmers as a whole must be net sellers of futures, to meet the demand from speculators. But an unusually risk tolerant farmer might be a net purchaser.

9. Rational Expectations Equilibrium and Information

In the model which I have just analysed the spot price is stochastic and differs from year to year, but the futures price is a constant, a function of the parameters of the model, including the mean and variance of $\tilde{a}$, the stochastic intercept in the spot demand function, which is by assumption the source of all the uncertainty.

I am now going to modify the model by assuming that dealers have information about $\tilde{a}$ in January when the futures market operates. I will look at three different information structures of increasing complexity, asking in each case how well the futures price reflects the information.

Example 1. *Symmetric Information*

Assume that

$$\tilde{a} = \tilde{I} + \tilde{e}. \tag{9.1}$$

$\tilde{I}$ and $\tilde{e}$ are independent scalar normal random variables, $E\tilde{I} = E\tilde{a}$, and $E\tilde{e} = 0$. As the sum of normal variables is normal, $\tilde{a}$ is still normal, $\text{var}\,\tilde{a} = \text{var}\,\tilde{I} + \text{var}\,\tilde{e}$. Assume also that all dealers, farmers and speculators observe $\tilde{I}$ each January. Conditional upon the information $\tilde{I}$ each dealer believes correctly that $\tilde{a}$ is a normal random variable whose mean $E(\tilde{a}|\tilde{I}) = \tilde{I}$ is random, whereas $\text{var}\,(\tilde{a}|\tilde{I}) = \text{var}\,\tilde{e}$ is not random. The model is unchanged, apart from the fact that beliefs about the mean of $\tilde{a}$ change from year to year. The rational expectations equilibrium can be calculated as before. Paralleling (8.12) and (8.14)–(8.18)

$$\text{var}[\tilde{p}_s|\tilde{I}] = b^{-2}\,\text{var}[\tilde{a}|\tilde{I}] = b^{-2}\,\text{var}\,\tilde{e} \tag{9.2}$$

$$E[\tilde{p}_s|\tilde{I}] = \theta^{\star-1}\phi^{\star}E(\tilde{a}|\tilde{I}) = \theta^{\star-1}\phi^{\star}\tilde{I} \tag{9.3}$$

where

$$\phi^{\star} = 1 + r + mc^{-1}b^{-2}\,\text{var}[\tilde{a}|\tilde{I}]\left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1} \tag{9.4}$$

$$\theta^{\star} = b\phi^{\star} + mc^{-1} \tag{9.5}$$

$$\tilde{p}_s = \theta^{\star-1}\phi^{\star}E(\tilde{a}|\tilde{I}) + b^{-1}[(\tilde{a} - E\tilde{a}|\tilde{I})] = \theta^{\star-1}\phi^{\star}\tilde{I} + b^{-1}\tilde{e} \tag{9.6}$$

and

$$\tilde{p}_f = \theta^{\star-1}E(\tilde{a}|\tilde{I}) = \theta^{\star-1}\tilde{I}. \tag{9.7}$$

These equations differ from (8.12) and (8.14)–(8.18) in two ways. Firstly, the terms relating to the unconditional distribution of $\tilde{a}$, $E\tilde{a}$

and var $\tilde{a}$ in the previous equations, have been replaced by the corresponding terms for the distribution conditional upon the information $E(\tilde{a}|\tilde{I})$ and var $E(\tilde{a}|\tilde{I})$. Secondly, the futures price $\tilde{p}_f$ is now a random variable rather than a constant.

The expressions $\theta^\star$ and $\phi^\star$ are not random because var $(\tilde{a}|\tilde{I})$ is not random. Thus, provided the numerical values of $\theta^\star$ and $\phi^\star$ are known, it is possible to infer and $E[\tilde{p}_s\ \tilde{I}]$ from $\tilde{p}_f$.

$$E[\tilde{p}_s|\tilde{I}] = \theta^{\star-1}\phi^\star\tilde{I} = \phi^\star\tilde{p}_f$$

and so the conditional distribution of $\tilde{p}_s$ given $\tilde{p}_f$ is normal

$$E[\tilde{p}_s|\tilde{p}_f] = E[\tilde{p}_s|\tilde{I}] = \phi^\star\tilde{p}_f \tag{9.8}$$

and

$$\text{var}[\tilde{p}_s|\tilde{p}_f] = \text{var}[\tilde{p}_s|\tilde{I}] = b^{-2}\ \text{var}\ (\tilde{a}|\tilde{I}) = b^{-2}\ \text{var}\ \tilde{e}. \tag{9.9}$$

Anyone knowing the numerical value of $\phi^\star$ would form the same conditional expectation of the spot price $\tilde{p}_s$ from the futures price $\tilde{p}_f$, as if he knew the information $\tilde{I}$. This observation is perhaps not very interesting in the context of this example in which, by assumption, all the dealers know $\tilde{I}$, but it is helpful in analysing the next two examples.

Example 2.

As in the previous example

$$\tilde{a} = \tilde{I} + \tilde{e}.$$

$\tilde{I}$ and $\tilde{e}$ are independent, and normal. $E\tilde{a} = E\tilde{I}$, and $E\tilde{e} = 0$. However, now only some of the dealers observe $\tilde{I}$. The others have no private information. In the rational expectations equilibrium the uninformed dealers will infer what information they can about the spot price from the futures price. If the futures price is completely efficient as an information transmitter the uninformed traders will trade as if they had the information.

This observation suggested to Radner (1979) and Grossman (1978) that models with asymmetric information could be analysed by considering the corresponding model in which the information is pooled and made available to all dealers (called a full communication equilibrium by Radner, an artificial economy by Grossman). If the futures price is a perfect transmitter of information in the rational expectations equilibrium of the original model, dealers' beliefs about the distribution of the spot price given the futures price in the original model will be the same in the full communications equilibrium as in the rational expectations equilibrium of the original model.

Observing this point, Radner and Grossman argued that the first step in analysing this type of model should be to examine the full communications equilibrium. This is much easier than looking at the rational expectations equilibrium with asymmetric information directly, because if dealers know all the information which could possibly be reflected in prices already they have no motive for using prices as information, so prices do not affect beliefs in the full communications equilibrium. Having characterized prices in the full communications equilibrium, ask what dealers' correct beliefs would be conditional on the full communications equilibrium prices. In particular ask whether the beliefs are the same as they would be if dealers know all the information. If they are it has been established that a rational expectations equilibrium exists in which beliefs, prices, supply and demand are the same as in the full communications equilibrium.

Consider the four-part definition of a rational expectations equilibrium in an asset market with asymmetric information. The first part refers to beliefs, the second to utility maximization given beliefs, the third to market clearing, and the fourth to correct beliefs. If the full communications equilibrium prices allow dealers to form precisely the same beliefs as if they had all the information, utility maximization leads to the same trades as in the full communications equilibrium. As the trades are the same, the market clears at the same prices. The beliefs generating the trades are correct. This is a rational expectations equilibrium. This argument breaks down if beliefs given the full communications equilibrium prices are not the same as beliefs given all the information. In this case if a rational expectations equilibrium exists prices transmit some but not all information.

In this example the full communications equilibrium is one in which all dealers observe $\tilde{I}$. This is precisely Example 1 where I have already argued that conditioning on the futures price alone leads dealers to the same beliefs as if they knew the information $\tilde{I}$. Thus the full communications equilibrium prices of Example 1 are also rational expectations equilibrium prices for Example 2. In this rational expectations equilibrium the futures price transmits all the information from the informed to the uninformed dealers.

Example 3. *Diverse Information*

I now generalize the information structure considerably. Suppose that each dealer observes a random information variable $\tilde{I}_i$. This may be a scalar or a vector, it may be constant, in which case it is effectively no information. The only restriction is that $(\widetilde{a}, \widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n)$ has a joint normal distribution. It seems an impossible task to ask a single price to aggregate all this diverse information, so that in the rational expectations equilibrium, dealers can trade as if they had all the information. Yet this is in fact so, owing to the following properties of normal random variables.

Lemma. *Conditional distributions of normal random variables*

If $[\widetilde{a}, \widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ has a joint normal distribution

$$\widetilde{I} = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$$

and

$$\widetilde{e} = \widetilde{a} = \widetilde{I}$$

then $\widetilde{I}$ and $\widetilde{e}$ are independent normal random variables, $E\,\widetilde{a} = E\widetilde{I}$, $E\widetilde{e} = 0 \text{var}\ \widetilde{a} = \text{var}\ \widetilde{I} + \text{var}\ \widetilde{e}$. The conditional distribution of $\widetilde{a}$ given $\widetilde{I}$ is the same as the conditional distribution of $\widetilde{a}$ given $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$. Both conditional distributions are normal, with mean

$$E(\widetilde{a}|\widetilde{I}) = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n] = \widetilde{I}$$

and variance

$$\text{var}(\widetilde{a}|\widetilde{I}) = \text{var}[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n] = \text{var}\ \widetilde{a} - \text{var}\ \widetilde{I} = \text{var}\ \widetilde{e}.$$

Proof. See Appendix.

This result shows that for the purposes of forming beliefs about $\widetilde{a}$ knowing $\widetilde{I} = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ gives the same information as knowing $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$. The conditional mean $\widetilde{I}$, a single number, aggregates perfectly all the diverse information. (It is a sufficient statistic for the information.)

The result can be used to compare two full communications equilibria, for the spot and futures market model. In the first equilibrium dealers observe the vector of random variables $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$. In the second they observe $\widetilde{I} = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$. In both equilibria the conditional distribution of $\widetilde{a}$ is normal, with the same mean and variance. Thus the equilibrium prices are the same. The equilibrium prices are the same. The equilibrium in which all dealers observe $\widetilde{I}$ is the equilibrium of the first model studied in this section. The prices in both equilibria are given by (9.2)–(9.7). In these equilibria, from (9.7)

$$\widetilde{p}_f = \theta^{\star -1}\widetilde{I} = \theta^{\star -1} = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n] \tag{9.10}$$

and from (9.2) and (9.3)

$$\text{var}[\widetilde{p}_s|\widetilde{p}_f] = \text{var}[\widetilde{p}_s|\widetilde{I}] = \text{var}[\widetilde{p}_s|[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] = b^{-2}\text{var}\widetilde{e} \quad (9.11)$$

$$E[\widetilde{p}_s|\widetilde{p}_f] = E[\widetilde{p}_s|\widetilde{I}] = E[\widetilde{p}_s|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] = \theta^{\star-1}\phi^{\star}\widetilde{I} = \phi^{\star}\widetilde{p}_f. \quad (9.12)$$

Conditioning only on the futures price dealers form the same beliefs about the spot price as they would if they know either $\widetilde{I} = E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n]$ or the entire information vector $\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n$. By the same argument as before, these must also be rational expectations equilibrium prices for the model in which dealer i observes information $\widetilde{I}_i$.

This is a much stronger result than before. It argues that a market price can not only transmit a single piece of information from one set of dealers to another, but also aggregate a large and diverse set of information perfectly.

10. The robustness of the international efficiency result

In the previous section I showed that in a simple futures market model the market price can aggregate diverse information so efficiently that each dealer's beliefs about the return on holding an asset (the spot price) given only its price are the same as if he had access to all the information to the market. He finds his own private information completely redundant.

The surprising result is not limited to futures markets. From a speculator's point of view a futures contract is one of many financial assets, others include shares and bonds issued by firms, and government securities. The original version of this model (Grossman 1976) considered a stock market. The stock lasts for one period, and pays a random gross return $\widetilde{R}$. An investor with wealth W_{i0} who buys x_i units of the stock at price p and invests $W_{i0} - px_i$ in a safe asset paying interest r, has final wealth

$$\widetilde{W}_i = (\widetilde{R} - p(1+r))x_i + W_{i0}(1+r).$$

The gross return $\widetilde{R}$ plays a role precisely analogous to the spot price in the futures market. If $\widetilde{R}$ is normally distributed and the investor has an exponential utility function $-\exp[-k_i\widetilde{W}_i]$ the argument used to derive the speculators' demand for futures yields the investors' demand for the stock

$$x_i = \frac{1}{k_i \text{var } \widetilde{R}}(E\widetilde{R} - p(1+r)). \quad (10.1)$$

If there are n investors and a fixed supply of the stock S, market clearing requires that

$$\sum_{i=1}^{n} \frac{1}{k_i \text{var } \widetilde{R}}(E\widetilde{R} - p(1+r)) = S. \qquad (10.2)$$

The stock and futures markets models are mathematically very similar, apart from the fact that the supply of the asset in the stock market is taken as exogenous.

Now suppose the investors have diverse information $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$, and $[\widetilde{R}, \widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ is joint normal. Experience with the futures market model suggests looking at the full communications equilibria, in which market clearing implies that

$$\sum_{i=1}^{n} \frac{1}{k_i \sigma^2}[E[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2 \ldots \widetilde{I}_n] - (1+r)\widetilde{p}] = S$$

where $\sigma^2 = \text{var}[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2 \ldots \widetilde{I}_n]$ so

$$[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2 \ldots \widetilde{I}_n] = \sigma^2 \left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1} S + (1+r)\widetilde{p}. \qquad (10.3)$$

Anyone knowing the numerical value of $\sigma^2 \left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1} S$ and $(1+r)$ could infer $[E[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ from the price $\widetilde{p}$, and would form the same beliefs about $\widetilde{R}$ as if he knew $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$. By a now familiar argument this implies that the full communications equilibrium is also a rational expectations equilibrium; the rational expectations equilibrium price aggregates the information perfectly.

Grossman wrote the paper embodying this result before he had the idea of using an artificial economy, or full communications equilibrium, to analyse the model. He had to use more complex arguments and was not able to prove such a general result. The paper was important firstly because it was the first satisfactory asset market model embracing risk aversion and asymmetric information, and secondly because Grossman pointed out a most important paradox. In Grossman's model, just as in the spot and futures market model, knowing the asset price renders dealers' private information redundant. If this information is costly no dealer has any incentive to gather the information, particularly if he knows that another dealer is using the same information. Yet if no one gathers the information it cannot be reflected in the price, which generates incentives to gather the information.

Grossman and Stiglitz (1980) resolve this paradox by modifying the model slightly. Suppose now that the asset supply is a normal random variable $\widetilde{S}$. The relationship between the full communications equilibrium price $\widetilde{p}$, $[E[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ and $\widetilde{S}$ is given by (10.3), modified only by replacing the constant S by random $\widetilde{S}$

$$[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n] = \sigma^2 \left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1} \widetilde{S} + (1+r)\widetilde{p}. \qquad (10.4)$$

Even if the numerical values of $\sigma^2 \left[\sum_{i=1}^{n} k_i^{-1}\right]^{-1}$ and $(1+r)$ are known it is impossible to infer $E[\widetilde{R}|\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n]$ from $\widetilde{p}$ because $\widetilde{S}$ is different each time the market operates. Conditioning on $\widetilde{p}$ does not yield the same information as conditioning on $\widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_n$. The full communications equilibrium is not a rational expectations equilibrium. (This is also true in the spot and futures market model, if farmers' output is uncertain, and dealers have information about both spot demand and output (Bray 1981).)

Grossman and Stiglitz calculate the rational expectations equilibrium for a version of the stock market model in which there are two groups of dealers. The informed dealers all observe the same information, on which they base their expectations. The uninformed dealers form their expectations on the basis of the price. The informativeness of the price increases as the proportion of informed dealers increases. In the absence of information costs the informed dealers have higher expected utility than the uninformed, because they are less uncertain of the asset return (its conditional variance is lower for the informed than the uninformed). If information is costly informed dealers may be better or worse off. If the proportion of informed dealers is large and the price conveys much of the information to the uninformed dealers they are likely to be worse off. If the proportion of informed dealers is small and the price conveys little information to the uninformed they are likely to be better off. Grossman and Stiglitz show that for each level of information costs there is an equilibrium proportion of informed dealers, so that the benefits of the information just balance the costs, and dealers are indifferent between being informed and uninformed. They derive a variety of interesting comparative static results from this model.

11. Existence of rational expectations equilibrium

Expectations play a crucial role in all the models which I have presented, as in many others. Whenever I have needed to close models by specifying expectations I have followed standard practice in postulating rational expectations. In each case I have been able to show that a rational expectations equilibrium exists by solving explicitly for the equilibrium. This is not always possible. Indeed in some examples, such as the one which follows, it can be shown that there is no set of prices and beliefs which satisfy Parts 3 and 4 of the definition. There is no rational expectations equilibrium.

The example is similar in form to that of Kreps (1977). It is a somewhat modified version of the spot and futures market model. For mathematical simplicity, assume that there is one farmer and one speculator. Each maximizes the expectation of a utility function $-exp(-\widetilde{W})$. The speculator believes the spot price $\widetilde{p}_s \sim N(\mu, \sigma^2)$. The interest rate $r = 0$. Arguing as before the speculators' excess demand for futures is

$$x_s = \frac{1}{\sigma^2}(\mu - p_f). \tag{11.1}$$

The farmer has a cost function for output $C(y) = sy + \frac{1}{2}y^2$. He also believes that $\widetilde{p}_s \sim N(\mu, \sigma^2)$. Utility maximization for the farmer implies that he sets output so $C'(y) = p_f$ or

$$y = p_f - s. \tag{11.2}$$

He hedges by selling y on the futures market, and in addition speculates by buying futures

$$x_f = \frac{1}{\sigma^2}(\mu - p_f). \tag{11.3}$$

Futures market clearing implies that $x_s + x_f = y$, or from (11.1)-(11.3)

$$\frac{2}{\sigma^2}(\mu - p_f) = p_f - s. \tag{11.4}$$

Spot demand is

$$D(p_s) = \widetilde{a} - \widetilde{p}_s$$

where $\widetilde{a}$ is a normal random variable, and var $\widetilde{a} = 1$. Spot market clearing implies that $D(\widetilde{p}_s) = y$, that is

$$\widetilde{a} - \widetilde{p}_s = p_f - s. \tag{11.5}$$

Equation (11.5) implies that

$$E\widetilde{p}_s = E\widetilde{a} - p_f + s \tag{11.6}$$

and

$$\text{var } \tilde{p}_s = \text{var } \tilde{a} = 1. \tag{11.7}$$

If the farmer and speculator are to form rational expectations $\mu = E\tilde{p}_s = E\tilde{a} - p_f + s$, and $\sigma^2 = \text{var } \tilde{p}_s = 1$. The futures market clearing condition (11.4) becomes

$$2(E\tilde{a} - p_f + s - p_f) = p_f - s$$

so

$$p_f = \frac{1}{5}(2E\tilde{a} + 3s). \tag{11.8}$$

Spot market clearing and rational expectations imply (11.6), which with (11.8) implies that

$$E\tilde{p}_s = \frac{1}{5}(3E\tilde{a} + 2s). \tag{11.9}$$

So far I have had no difficulty in calculating the rational expectations equilibrium, but introducing asymmetric information can cause complications. Suppose that there are only two sorts of weather, good and bad. The farmer observes the weather; the speculator does not. If the weather is good $E\tilde{a} = 5/4$ and $s = 1/6$. If it is bad $E\tilde{a} = 1$ and $s = 1/3$.

There are only two possibilities, either the futures price is different in different weather or it is not. If the futures price is different the speculator can infer the weather from the price. Trades and prices will be the same as if both farmer and speculator knew the weather. In this case in good weather $E\tilde{a} = 5/4$, $s = 1/6$, from (11.8) $p_f = 3/5$ and from (11.9) $E\tilde{p}_s 49/60$. In bad weather $E\tilde{a} = 1$, $s = 1/3$, from (11.8) $p_f = 3/5$ and from (11.9) $E\tilde{p}_s 44/60$. Thus the futures price is the same in both weather, contradicting the supposition that it was different.

The alternative supposition is that the futures price is the same whatever the weather, in which case the speculator's demand will be the same. If the farmer has rational expectations his excess demand for futures will be using (11.2), (11.3) and (11.6), and recalling that $\sigma^2 = 1$ and $\mu = E\tilde{p}_s$,

$$\begin{aligned} x_f - y &= (E\tilde{p}_s - p_f) - (p_f - s) = E\tilde{p}_s + s - 2p_f \\ &= E\tilde{a} + 2s - 3p_f. \end{aligned}$$

In good weather $E\tilde{a} + 2s = 19/12$, in bad weather $E\tilde{a} + 2s = 5/3$. If p_f is independent of the weather, the speculator's demand for futures is independent of the weather, but the farmer's is not. The futures market cannot clear at the same price in both weathers. This exhausts the possibilities. In this example the assumption of market clearing and rational expectations are logically inconsisent. There is no rational expectations equilibrium.

In defining a rational expectations equilibrium for an asset market model in §3 I argued that the market clearing condition induces a mapping from the beliefs people hold to the correct beliefs. This is an almost universal feature of models with exptectations; it crops up, for example, in equation (8.11) which gives the correct expected spot price $E\tilde{p}_s$, as a function of the subjectively held expectation μ. A rational expectations equilibrium is a fixed point of this mapping. Fixed point theorems give conditions under which mappings have fixed points; notably continuity. The non-existence problems for rational expectations models with asymmetric information stem from discontinuities in the mapping, where a small change in prices can induce a large change in the information which can be inferred from them. In the example, if prices are identical in both weathers, the speculator cannot infer the weather, but if they are very slightly different he can.

Checking that an equilibrium exists is an essential preliminary to using a model; assuming that an equilibrium exists and arguing from there, can yield no valid conclusions if in fact no equilibrium exists. Knowing the circumstances under which a model has an equilibrium puts logical limits on the range of applicability.

Existence problems are attacked from two directions, existence theorems and non-existence examples. Existence theorems establish that under certain conditions, typically conditions on preferences, technology, and the structure of transactions and information, an equilibrium exists. For some special models equilibrium can be shown to exist by calculating the equilibrium, but in general the problem is attacked indirectly, often using fixed point theorems which establish that a set of equations has a solution, but not what the solution is. Non-existence examples show that in certain cases no equilibrium exists. These examples are helpful because they show certain conjectured general existence results cannot be valid; a claim that *all* models of a certain type have an equilibrium is wrong if a single such model has no equilibrium, just as a single black swan is enough to invalidate the claim that all swans are white.

The non-existence example which I demonstrated earlier is not robust; a small change in the parameters of the model would allow an equilibrium to exist; non-existence is a freak eventuality. Radner (1979) studies a much more general asset market model which shares two features with this example. In both models there are only a finite number of different possible information signals. In the example there are two, good weather or bad weather. In Radner's model there may be a large but finite number of different signals received by a finite number of individuals. The vector of joint signals can only take a finite number of different values. In Radner's model, as in my example, there may be no rational expectations equilibrium. Radner shows rigorously that equilibrium exists generically. Generic existence is defined precisely in the paper; the idea which it captures is that whilst equilibrium may fail to exist in some special cases, almost any perturbation of the model will restore existence. Radner's proof proceeds by considering the full communications equilibrium in which dealers pool all their information signals before trading. The price vector in the full communications equilibrium $\tilde{p}$ is a function of the joint signal $\tilde{s}$, $\tilde{p} = p(\tilde{s})$. If the price vector is different whenever any element in the signal is different, the price reveals the signal, the full communications equilibrium is a rational expectations equilibrium, in which prices fully reveal the information.

The crucial question is whether the map from the signals into prices is invertible. There are a finite number, m, of signals, whereas prices can be any vector in $\mathbb{R}^{n+}$, so there are an infinity (indeed a continuum) of different possible prices. Radner's result confirms the intuition that if the utility functions generating demand are reasonably well behaved, the map from signals to prices fails to be invertible only in special circumstances, in which case a small perturbation of the model restores invertibility.

The assumption that there are a finite number of different possible signals plays a crucial role in this invertibility argument. If there is a continuum of different possible signals the argument may break down. Jordan and Radner (1982) devise an example with informed and uninformed dealers and one relative price. The informed dealer observes a signal $\tilde{s}$ in [0,1]. Given the price, the informed dealer's demand changes with the signal, if there are two different signals $s_i \neq s_2$, with $p(s_i) = p(s_2)$ the informed dealer's demand is different for the two signals, but the uninformed dealer who observes only the price has the same demand. The market cannot clear at the same price for both s_1 and s_2. On the other hand, if the function is invertible the uninformed dealer can infer s from p, the prices are the same as in the full communications equilibrium. But Jordan and Radner show that the full communications equilibrium price function has the form shown in Fig. 2, and is not invertible. This is a robust example; changing the parameters of the model changes the price function a little, but does

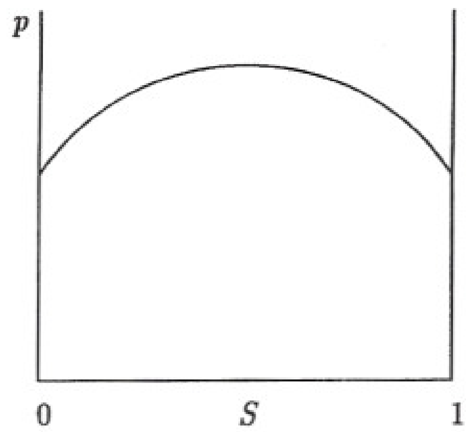

Figure 2

not make it invertible.

The importance of invertibility for the existence of rational expectations equilibria in which prices reveal all the information suggests that the relative dimensions of the signal space and the price space may be important. This is confirmed by Allen (1982) who shows that if the dimension of the signal space is less than the dimension of the price space a fully revealing rational expectations equilibrium exists generically. Jordan (1983) shows that if the dimension of the price space is higher than the dimension of the signal space rational expectations equilibrium exists, generically, but is not fully revealing.

The literature on fully revealing equilibria is concerned with equilibria in which dealers can infer the entire information signal from the prices. This is sufficient to enable them to form the same expectations as if they saw the signal. But it is not necessary; dealers want to know about a vector of asset returns $\tilde{R}$. If $\tilde{R}$ and the information $\tilde{I}$ are joint normal, knowing $E(\tilde{R}|\tilde{I})$ tells them as much as knowing $\tilde{I}$. The vector $E(\tilde{R}|\tilde{I})$ has the same dimension as $\tilde{R}$, the number of risky assets. This may be much lower than the dimension of $\tilde{I}$. Grossman (1978) uses this result to analyse a stock market model in which returns are normal, and dealers care only about the mean and variance of return. By applying the capital asset pricing model Grossman shows that, provided the market portfolio is not a Giffen, good dealers can infer $E(\tilde{R}|\tilde{I})$ from the information, and so a rational expectations equilibrium exists in which dealers trade as if they had all the information. Grossman also exploits the properties of normal random variables in his paper on futures markets (1977), showing how they can act to transmit information.

The existence of rational expectations equilibrium in asset markets is an attractive and challenging problem for mathematical economists.

A more sophisticated discussion and further references are in Radner (1982), which surveys the literature on 'Equilibrium under uncertainty', and by Jordan and Radner (1982) which introduces a symposium issue of the *Journal of Economic Theory*, on 'Rational expectations in microeconomic models', which includes a number of related papers. More recent work on the matter includes Jordan (1982*a*, 1982*b*), Allen (1983) and Anderson and Sonnenschein (1985).

12. Evaluating the models

A model is a simplified, stylized description of certain aspects of the economy. It omits many details in order to concentrate on certain features and their interrelationships. One of the major objectives of modelling is often to show that the description is logically consistent by demonstrating that an equilibrium exists, an issue which I have discussed at some length. If a model is to be used as a basis for saying something about real economies, logical consistency is essential; even grossly unrealistic models may be useful in establishing logical limits to rhetoric. But it is obviously desirable that a model be a correct, as well as a consistent, description.

Unfortunately there is no clear and universally applicable criterion for the correctness of models. Any model omits details, abstracts and simplifies. Reality is too complicated to be thought about in totality. Assumptions in economic models are most unlikely to be completely adequate descriptions of behaviour. The question to ask is whether they are plausible enough to generate implications which say something about the aspects of reality with which the model is concerned. This is inevitably a matter of judgement, and must often depend upon the use to which a model is being put.

The three major assumptions made in the financial market models which I described are that markets clear, that agents are price-takers and that they have rational expectations. These assumptions are very widely made; they are also central to the 'new-classical' macroeconomics (Begg 1982*a*). Market clearing and price-taking seem in general quite plausible for financial markets, where prices move readily, there is little evidence of sustained excess supply and demnd, and a large number of traders.

The rational expectations hypothesis can be stated loosely, that people do not make systematic mistakes in forecasting; more precisely, people's subjective beliefs about probability distributions correspond to the objective probability distributions. Employing the rational expectations hypothesis imposes two logical requirements, that objective probability distributions exist, and that a rational expectations equilibrium exists. In constructing a model an economist creates the objective probability distributions, but these models can only be applied to situations where the ditributions could in principle at least be derived from data. This requires that the structure and parameters of the economy are in some way constant through time. Rational expectations models describe long run stationary equilibria.

One important criticism of the rational expectations hypothesis is that it assumes that agents know too much. Conider the spot and futures market model with asymmetric information. In rational equilibrium the uninformed dealers believe correctly that the conditional distribution of the spot price, given the futures price is normal, has conditional mean given by (9.8) $E[\widetilde{p}_s|\widetilde{p}_f] = \phi^\star \widetilde{p}_f$ and a constant conditional variance. All they need to know is the fact of normality, and the numerical value of $\phi^\star$ and $\operatorname{var}[\widetilde{p}_s|\widetilde{p}_f]$. The uninformed dealers do not have to know the structure of the model, just two parameters of the reduced form. Further, by observing the markets operating in rational expectations equilibrium for a number of years, the numbers $\phi^\star$ and $\operatorname{var}[\widetilde{p}_s|\widetilde{p}_f]$ could be estimated by standard statistical techniques. Apparently it is quite easy to learn how to form rational expectations.

In financial markets there are very large amounts of money at stake; and there is every incentive to apply the considerable abilities and resources of professional investors to make the best possible forecasts. However, the argument that is is easy or even possible to learn how to form rational expectations by applying standard statistical techniques is misleading. Economists are interested in expectations because they believe that expectations affect what happens. This belief is reflected in the models; if agents in these models do not have rational expectations, the model behaves differently from the rational expectations equilibrium. In §7 I defined a rational expectations equilibrium as a fixed point of the mapping from subjectively held beliefs into 'correct beliefs' induced by the market clearing condition. Outside rational expectations equilibrium subjective beliefs differ from both correct beliefs, and the rational expectations equilibrium beliefs. For example, in the spot and futures market model in rational expectations equilibrium, dealers believe that $E[\widetilde{p}_s|\widetilde{p}_f] = \phi^\star \widetilde{p}_f$. If dealers believe that $E[\widetilde{p}_s|\widetilde{p}_f] = \phi^\star \widetilde{p}_f$ where $\phi \neq \phi^\star$, the correct conditional expectation will be of the form $E[\widetilde{p}_s|\widetilde{p}_f] = \hat{\phi}\widetilde{p}_f$ where $\hat{\phi} \neq \phi$, the expectation is incorrect, and $\hat{\phi} \neq \phi^\star$, the correct expectation is not the same as in the rational expectations equilibrium. Changing to the 'correct' expectation formation rule $E[\widetilde{p}_s|\widetilde{p}_f] = \hat{\phi}\widetilde{p}_f$ changes the behaviour of the model, and thus rule becomes incorrect. The obvious question to ask is whether repeated changes of the expectation formation rule ultimately lead to a rational expectations equilibrium. Is it possible to describe a plausible learning proces which ultimately yields rational expectations? The answer depends upon how 'plausible' is understood. One possibility is to insist that agents learn using correctly specified Bayesian models. David Kreps and I argue elsewhere (Bray and Kreps 1986) that it is in fact not plausible, because it in effect assumes a more elaborate and informationally demanding form of rational expectations equilibrium; for example, in the spot and futures market they regress $\widetilde{p}_s$ on $\widetilde{p}_f$ using ordinary least squares, and use the estimated regression coefficients in forecasting $\widetilde{p}_s$ from $\widetilde{p}_f$. In Bray (1982) I studied this procedure for the model of Example 2 where there are uninformed dealers, and informed dealers all of whom have the same information. I found that provided the uninformed dealers did not form too large a proportion of the market, the model would eventually converge to its rational expectations equilibrium. Bray (1983)[3] and Bray and Savin (1984) study similar econometric learning processes for a simple macroeconomic model and a version of the cobweb model. In both these models if the parameters of the supply and demand functions have the usual signs agents eventually learn how to form rational expectations. In all these examples agents are estimating misspecified economic models, so convergence to rational expectations equilibrium is not based on standard theorems on the asymptotic properties of estimators, is somewhat surprising, and hard to prove. Convergence to rational expectations equilibrium may be slow, and takes place only if the parameters of the model lie in a certain range. Although many of the examples which have been studied converge in economically plausible circumstances, there is no general theory which establishes that convergence will always take place.

Expectations are important for economics; they crop up unavoidably in considering a vast range of issues. The enormous virtue of the rational expectations hypothesis is that it gives a simple, general and plausible way of handling expectations. It makes it possible to formulate and answer questions, for example, on the efficiency of markets as transmitters of information, which would otherwise by utterly intractable. All recent progress on the economics of information is built on the rational expectations hypothesis.

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[3] Bray (1983) is much the shortest and simplest of these papers on learning, and the best introduction to the issues as I see them. Bray and Savin (1984) contains computer simulations which shed light on the rates of convergence and divergence, and discusses the relationship between this work, and time-varying parameter models in econometrics. Related literature is surveyed briefly in Blume, Bray and Easley (1982). Bray and Savin (1984) contains more recent references.

Consider for a moment the alternative hypotheses. One possibility is that agents use a simple forecasting rule which generates systematic mistakes. In any application it is necessary to specify the rule, for example adaptive expectations. If there is good evidence that people do forecast in this way, this is attractive, but it seems implausible that in the long run in a stable environment they will fail to notice their mistakes and modify the rule. Another alternative is to try to model the dynamics of the learning process. At present this seems to make for models which are too complicated and mathematically difficult to use for addressing most questions. Rational expectations equilibrium is a way of avoiding many difficult dynamic issues; if an issue is intractable in the current state of knowledge, circumventing it is probably the most fruitful research strategy.

Another alternative is to rely on survey data for expectations. Where possible this may be valuable in empirical work, if not very helpful for theorists.

A further alternative is to follow Keynes and argue that expectations cannot be described as probability distributions; they are volatile, and not susceptible to formal description. This makes it impossible to incorporate expectations explicitly into formal models, except by treating them as exogenous. Begg (1982*b*) argues that this is Keynes' strategy in the General Theory and is followed in traditional textbook treatments of Keynesian theory. In some cases I think this is an entirely defensible, indeed attractive strategy for modelling short-term events. The danger is that if expectations are unobservable, inexplicable, exogenous and volatile it leaves the model with no predictive and very little explanatory power as anything can be attributed to a shift in expectations. The rational expectations hypothesis also postulates unobservable expectations, but otherwise in total opposition to Keynes treats expectations as explicable, exogenous, and stable (unless the underlying model changes in which case expectations change appropriately). In medium to long term models the extreme rational expectations hypothesis is more attractive than the extreme exogenous expectations hypothesis. There is currently no generally acceptable intermediate hypothesis. Note that although Keynes himself would probably shudder if he knew, there is no reason why rational expectations should not be incorporated into 'Keynesian' models, which would have quite different properties from the 'new-classical' rational expectations models (see Begg 1982*b*).

The rational expectations hypothesis seems at present much the most satisfactory generally applicable hypothesis on expectations formation. But it must be remembered that rational expectations models describe long run equilibria, on the assumption that the dynamics induced by learning eventually converge to rational expectations equilibrium. We have no good reason to believe that this assumption is always, or even often, valid.

I have discussed the assumptions of the financial market models at some length. The other criteria for the correctness of the models as descriptions is to look at implications of the models, and compare them with data. There are two sources of data, experimental data from laboratory situations, and empirical data from real markets. Ultimately the objective is to understand real markets, but laboratory data generated by setting up a market with groups of students, enables the experimenter to control and design the experiment, eliminating the host of extraneous factors which affect real market data.

Plott and Sunder (1982) set up a series of asset markets with informed and uninformed traders. The return on the asset depended on which of two or three states of the world occurs. The informed traders all had the same piece of information, in most cases telling them which state of the world had occurred. Plott and Sunder calculated two prices for each market, firstly the rational expectations equilibrium price in which the uninformed dealers inferred as much as possible from the price, secondly the prior information price in which the uninformed dealers traded only on the basis of their prior information. Although the rational expectations model was not a perfect fit, prices did show a tendency to move towards their rational expectations equilibrium level. Plott and Sunder interpret the data as supporting the rational expectations rather than the prior information model.

Real market data has been used to test the efficient markets hypothesis, that using information in addition to the current price of an asset does not make for better predictions, the market price efficiently aggregates all the information. Three different forms of the hypothesis have been conidered, the weak form, considering the information in past prices, the semi-strong form, considering more general publicly available information, and the strong form, considering private information. The empirical literature is vast; Brealey (1983) provides a very readable introduction, and numerous references. Broadly the literature supports the weak and semi-strong forms of the efficient markets hypothesis, but private information does seem to give some advantage. The efforts of numerous academic investigators have failed to uncover a rule for forecasting market prices in order to manage a portfolio which does significantly better than holding a fixed, well-diversified portfolio. These results are consistent with the theoretical models which I have been describing and can be taken as support for the application of the rational expectations hypothesis to financial markets.

13. Further questions

These models answer some questions, but provoke others. Many of the models consider asset markets in isolation, taking the return generated by the asset as exogenous. (The spot and futures market model is an exception.) But financial markets are part of a larger sysem. One of their major functions is to enable enterprises to spread, and share risk, with consequences for output, investment and employment. It now appears that the markets may also have a role as transmitters of information. The ramifications of this role are not understood, but may be investigated using techiques similar to those which I have described.

Another set of open questions concern the mechanism of price formation. In these models price is a function of information, for example in the spot and future market model, where dealers have diverse information, the futures price $\tilde{p}_f = \theta^{\star-1}E[\tilde{a}|\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n]$, (9.10), where $\theta^\star$ is a parameter, and $\tilde{I}_i$ agent i's information, a normal random variable. As the information varies from year to year the price varies. If the dealers have diverse information no individual dealer can check that the price is at the correct level given all the information. If a dealer thinks that the futures price is high or low given his private information, he can only conclude that other dealers have different information which leads them to expect a high or low spot price. Any numerical value of $\tilde{p}_f$ can clear the market; it is far from clear what pushes $\tilde{p}_f$ to its correct value. (This point is originally due to Beja 1976.)

Universal price-taking is of course a convenient fiction. People set prices, unilaterally, by auction procedures, or by haggling. If there is a very limited range of prices at which goods can be sold, price-taking is a good approximation. It may be necessary to consider the detailed mechanics of price making, the activities of brokers, jobbers and market makers, to understand some aspects of the determination of prices in asset markets. In discussing their experimental results, Plott and Sunder suggest that some of the information is transmitted by the oral auction process which thee use, including unaccepted bids and offers. If this is so it provides an additional reason for looking at the institutional details of market structure.

The models which I cite have a very stark, simple, time structure; things happen at only two dates. In practice many financial markets operate repeatedly; the same asset is traded at a large number of dates, indeed trade may best be modelled as a continuous time process. There is a literature on continuous time models of financial markets (e.g. Black and Scholes 1973; Merton 1973), but this literature takes no account of informational asymmetries. Continuous time models with asymmetric information are attractive means of investigating the rate at which markets disseminate information, although they may pose formidable technical difficulties. There is certainly a case for looking at a richer temporal structure than has been considered up to now.

Appendix

Proof of Lemma. *Conditional distributions of normal random variables*

Anderson (1958) shows that

$$\tilde{I} = E[\tilde{a}|\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n] = E\tilde{a} + \sum_{ay} \sum_{yy}^{-1} (\tilde{y} - E\tilde{y}) \tag{A.1}$$

where $\tilde{y}$ is notation for the vector $[\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n]$, $\sum_{ay} = \mathrm{cov}(\tilde{a}, \tilde{y})$, $\sum_{yy} = \mathrm{var}(\tilde{y})$. Equation (A.1) implies that $\tilde{I}$ is a linear function of $[\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n]$. As linear functions of normal notation variables are normal, $\tilde{I}$ and $\tilde{e} = \tilde{a} - \tilde{I}$ are normal.

$$\mathrm{cov}(\tilde{e}, \tilde{y}) = \mathrm{cov}\left[\tilde{a} - E\tilde{a} - \sum_{ay} \sum_{yy}^{-1} (\tilde{y} - E\tilde{y}), \tilde{y}\right]$$

$$= \sum_{ay} - \sum_{ay} \sum_{yy}^{-1} \sum_{yy} = 0.$$

Thus $\tilde{e}$ and $\tilde{y}$ are uncorrelated and, as they are normal independent. Since $\tilde{I}$ is a linear function of $\tilde{y}$, $\tilde{I}$ and $\tilde{e}$ are uncorrelated, that is

$$\mathrm{cov}(\tilde{I}, \tilde{e}) = \mathrm{cov}(\tilde{I}, \tilde{a} - \tilde{I}) = 0 \tag{A.2}$$

and so $\tilde{I}$ and $\tilde{e}$ are independent.

From (A.1)

$$E\tilde{a} = E\tilde{I} \tag{A.3}$$

and so $E\tilde{e} = E\tilde{a} - E\tilde{I} = 0$. As $\tilde{I}$ and $\tilde{e}$ are independent

$$\mathrm{var}\ \tilde{a} = \mathrm{var}\ (\tilde{I} + \tilde{e}) = \mathrm{var}\ \tilde{I} + \mathrm{var}\ \tilde{e}.$$

As $\widetilde{I}$ is a function of $\widetilde{I}_1, \dots, \widetilde{I}_n$ and $\widetilde{e}$ is independent of $\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n$, the conditional distribution of $\widetilde{a} = \widetilde{I} + \widetilde{e}$ given $\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n$ is normal (as $\widetilde{e}$ is normal), with mean

$$E[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] = E[\widetilde{I}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] + E(\widetilde{e}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n]$$
$$= \widetilde{I} + E\widetilde{e} = \widetilde{I} = E(\widetilde{a}|\widetilde{I})$$

and

$$\text{var}[\widetilde{a}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] = \text{var}[\widetilde{e}|\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n] = \text{var } \widetilde{e} = \text{var}(\widetilde{a}|\widetilde{I}).$$

It can be shown that the conditional expectation of $\widetilde{a}$ given $\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n$ is the unique linear function of $\widetilde{I}$ of $\widetilde{I}_1, \widetilde{I}_2, \dots, \widetilde{I}_n$ satisfying (A.2) and (A.3). These equations characterize the conditional expectation of one normal random variable given another. (See Bray (1981) for an application of this fact.)

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